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LETTER TO THE EDITOR

Anisotropy and restricted universality of critical phenomena

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Abstract
We study the φ^4 lattice model with anisotropic short-range interactions. We show that the nonuniversal finite-size scaling functions of the anisotropic φ^4 model in a d-dimensional rectangular geometry are determined by the universal finite-size scaling functions of the isotropic φ^4 theory in a d-dimensional parallelepiped. Predictions are made for the Binder cumulant. In the bulk limit two-scale factor universality is absent in those universal critical-point amplitude relations that involve the correlation length.

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The concept of universality plays a fundamental role in the theory of critical phenomena of bulk and confined systems [1]. It is widely believed that, for systems with short-range interactions in $d < 4$ dimensions, the scaling functions and various critical-point amplitude ratios are independent of microscopic details. Recently, it was found within φ^4 continuum theory [2] that for anisotropic systems confined to rectangular geometries, the finite-size scaling functions depend on microscopic coupling constants and that this nonuniversal dependence could not be eliminated by a rescaling of the confining lengths. In particular it was concluded that, for rectangular geometries, the anisotropy effect cannot be inferred from the knowledge of finite-size scaling functions of isotropic systems of the same universality class. Very recent Monte Carlo (MC) data by Selke and Shchur [3] for the Binder cumulant [4] of the two-dimensional anisotropic Ising model on a square lattice support the analysis of [2]. Also in MC simulations for the three-dimensional anisotropic Ising model on a cubic lattice [5], a nonuniversal variation of the Binder cumulant was found (see also [6]).

In the present letter we study the problem of anisotropy on the level of the φ^4 *lattice* theory. We show that exact relations exist between nonuniversal critical phenomena in anisotropic lattice systems with rectangular geometries and universal critical phenomena in isotropic systems with *non-rectangular* (parallelepiped) geometries whose angles and edges are determined by the anisotropy parameters of the original lattice. Similar relations exist between other geometries. The crucial point is a rescaling of lengths *along the principle axes of the anisotropic system* rather than in the direction of the edges of the rectangular shape or in the direction of the symmetry axes of the original lattice. We present predictions for the Binder cumulant that can be tested by MC simulations. In the bulk limit we show the absence of two-scale factor universality [1] of anisotropic systems in the context of two universal amplitude relations introduced by Tarko and Fisher [7]. Our analysis demonstrates that universality of critical phenomena in bulk and confined systems is valid only in a restricted sense if noncubic anisotropies are present and that it is necessary to distinguish between isotropic and anisotropic systems within a universality class.

We start from the O(n) symmetric d-dimensional φ^4 lattice model with the Hamiltonian

$$H_{\mathrm{box}}^{\mathrm{aniso}} = v\left[\sum_{i=1}^{N}\left(\frac{r_0}{2}\varphi_i^2 + u_0(\varphi_i^2)^2 - h\varphi_i\right) + \sum_{i,j=1}^{N}\frac{K_{i,j}}{2}(\varphi_i-\varphi_j)^2\right] \tag{1}$$

where φ_i are n-component vectors on N lattice points $\mathbf{x}_i \equiv (x_{i1}, x_{i2}, \ldots, x_{id})$. We consider a simple-orthorhombic lattice with lattice constants $a_1, a_2, \ldots, a_d$ in a rectangular box with a volume $V = L_1 L_2 \cdots L_d = vN$, where $v = a_1 a_2 \cdots a_d$ is the volume of the elementary cell. Non-orthorhombic anisotropies may arise through the couplings $K_{i,j} = K_{j,i}$. They manifest themselves on macroscopic length scales via the dimensionless second moments

$$A_{\alpha\beta} = A_{\beta\alpha} = N^{-1}\sum_{i,j=1}^{N}(x_{i\alpha}-x_{j\alpha})(x_{i\beta}-x_{j\beta})K_{i,j}. \tag{2}$$

They are independent of the boundary conditions and geometry of the system. We assume short-range interactions and periodic boundary conditions. In terms of the Fourier components $\hat{\varphi}(\mathbf{k}) = v\sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}_j}\varphi_j$ and $\hat{K}(\mathbf{k}) = N^{-1}\sum_{i,j=1}^{N} K_{i,j}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}$ the Hamiltonian reads

$$\begin{aligned} H_{\mathrm{box}}^{\mathrm{aniso}} &= V^{-1}\sum_{\mathbf{k}}\frac{1}{2}[r_0 + \delta\hat{K}(\mathbf{k})]\hat{\varphi}(\mathbf{k})\hat{\varphi}(-\mathbf{k}) - h\hat{\varphi}(\mathbf{0}) \\ &\quad + u_0 V^{-3}\sum_{\mathbf{kpq}}[\hat{\varphi}(\mathbf{k})\hat{\varphi}(\mathbf{p})][\hat{\varphi}(\mathbf{q})\hat{\varphi}(-\mathbf{k}-\mathbf{p}-\mathbf{q})], \end{aligned} \tag{3}$$

where $\delta\hat{K}(\mathbf{k}) = 2[\hat{K}(\mathbf{0}) - \hat{K}(\mathbf{k})]$ has the long-wavelength form

$$\delta\hat{K}(\mathbf{k}) = \sum_{\alpha,\beta=1}^{d} A_{\alpha\beta}k_\alpha k_\beta + \sum_{\alpha,\beta,\gamma,\delta}^{d} B_{\alpha,\beta,\gamma,\delta}k_\alpha k_\beta k_\gamma k_\delta + O(k^6). \tag{4}$$

The summations $\sum_{\mathbf{k}}$ run over discrete vectors $\mathbf{k} \equiv (k_1, k_2, \ldots, k_d)$ with Cartesian components $k_\alpha = 2\pi m_\alpha / L_\alpha, m_\alpha = 0, \pm 1, \pm 2, \ldots, \alpha = 1, 2, \ldots, d$ in the range $-\pi/a_\alpha \leqslant k_\alpha < \pi/a_\alpha$. These values represent a simple-orthorhombic lattice in $\mathbf{k}$ space. The second moments $A_{\alpha\beta}$ at $O(k^2)$ affect the asymptotic critical behaviour whereas the terms of $O(k^4)$ yield only corrections (provided that $\det \mathbf{A} > 0$).

In [2] a transformation was introduced that brings the anisotropic gradient term of the continuum φ^4 model into an isotropic form. Here we apply this transformation to the φ^4 lattice model and provide a geometrical interpretation of the changes of the lattice structures in real space and in wave-vector space as well as of the change of the confining shape. We perform a rotation and a rescaling of the $\mathbf{k}$ vectors, $\mathbf{k}' = \boldsymbol{\lambda}^{1/2}\mathbf{U}\mathbf{k}$, such that the $O(k^2)$ term of $\delta\hat{K}(\mathbf{k})$ is transformed into an isotropic form,

$$\delta\hat{K}(\mathbf{U}^{-1}\boldsymbol{\lambda}^{-1/2}\mathbf{k}') = \sum_{\alpha,\beta=1}^{d} A'_{\alpha\beta}k'_\alpha k'_\beta + \sum_{\alpha,\beta,\gamma,\delta}^{d} B'_{\alpha,\beta,\gamma,\delta}k'_\alpha k'_\beta k'_\gamma k'_\delta + O(k'^6) \tag{5}$$

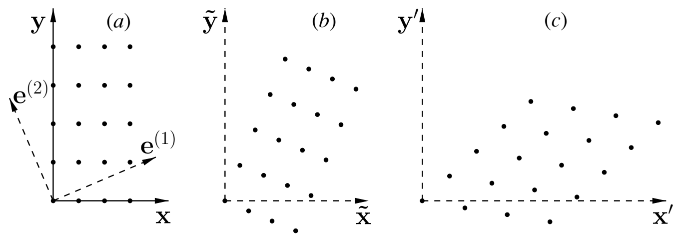

Figure 1. Lattice points of the φ^4 lattice model, the dashed lines indicate the principle axes. (*a*) Original lattice points $\mathbf{x}_j$, (*b*) rotated lattice points $\tilde{\mathbf{x}}_j = \mathbf{U}\mathbf{x}_j$, (*c*) rescaled lattice points $\mathbf{x}'_j = \boldsymbol{\lambda}^{-1/2}\tilde{\mathbf{x}}_j = \boldsymbol{\lambda}^{-1/2}\mathbf{U}\mathbf{x}_j$.

with $A'_{\alpha\beta} = (\boldsymbol{\lambda}^{-1/2}\mathbf{U}\mathbf{A}\mathbf{U}^{-1}\boldsymbol{\lambda}^{-1/2})_{\alpha\beta} = (\boldsymbol{\lambda}^{-1/2}\boldsymbol{\lambda}\boldsymbol{\lambda}^{-1/2})_{\alpha\beta} = \delta_{\alpha\beta}$. The orthogonal matrix $U_{\alpha\beta} = e_\beta^{(\alpha)}$, $(\mathbf{U}^{-1})_{\alpha\beta} = e_\alpha^{(\beta)}$ is determined by the d eigenvectors $\mathbf{e}^{(\alpha)}$ of the symmetric matrix $\mathbf{A}$ whose Cartesian components are denoted by $e_\beta^{(\alpha)}$. They satisfy the eigenvalue equations $\mathbf{A}\mathbf{e}^{(\alpha)} = \lambda_\alpha \mathbf{e}^{(\alpha)}$, $\mathbf{e}^{(\alpha)} \cdot \mathbf{e}^{(\beta)} = \delta_{\alpha\beta}$. The matrix $\mathbf{U}$ diagonalizes the matrix $(A_{\alpha\beta}) \equiv \mathbf{A}$ according to $\mathbf{U}\mathbf{A}\mathbf{U}^{-1} = \boldsymbol{\lambda}$ with diagonal elements $\lambda_\alpha > 0$. In real space the transformed second moments $A'_{\alpha\beta}$ are expressed as

$$A'_{\alpha\beta} = N^{-1} \sum_{i,j=1}^{N} (x'_{i\alpha} - x'_{j\alpha})(x'_{i\beta} - x'_{j\beta}) K_{i,j} = \delta_{\alpha\beta} \tag{6}$$

where the transformed lattice points are $\mathbf{x}'_\mathbf{j} = \boldsymbol{\lambda}^{-1/2}\mathbf{U}\mathbf{x}_\mathbf{j}$. This transformation leaves the scalar product $\mathbf{k}' \cdot \mathbf{x}'_\mathbf{j} = \mathbf{k} \cdot \mathbf{x}_\mathbf{j}$ invariant.

The geometrical interpretation is as follows (see figure 1). Because of the linearity of the coordinate transformation the planar surfaces remain planar and pairs of opposite surfaces remain parallel but the edges are, in general, no longer perpendicular to each other. Thus the d-dimensional box is distorted into a d-dimensional *parallelepiped*. More specifically, the directions of the eigenvectors representing the principle axes of the anisotropic system are, in general, not parallel to the symmetry axes of the orthorhombic lattice (figure 1(*a*), dashed lines). In the first step the orthorhombic lattice is described relative to the principle axes in terms of rotated coordinates $\tilde{\mathbf{x}}_j = \mathbf{U}\mathbf{x}_j$ (figure 1(*b*)). In this coordinate system the matrix of second moments

$$\tilde{A}_{\alpha\beta} = N^{-1} \sum_{i,j=1}^{N} (\tilde{x}_{i\alpha} - \tilde{x}_{j\alpha})(\tilde{x}_{i\beta} - \tilde{x}_{j\beta}) K_{i,j} = \lambda_\alpha \delta_{\alpha\beta}. \tag{7}$$

is diagonal, but with different diagonal elements λ_α. Subsequently, the lattice is distorted to a d-dimensional non-rectangular lattice with coordinates $\mathbf{x}'_j = \boldsymbol{\lambda}^{-1/2}\tilde{\mathbf{x}}_j$ by a rescaling of lengths *along the directions of the principle axes* (figure 1(*c*)). Thereby the orthorhombic elementary cells are distorted to parallelepipeds with the volume $v' = (\det \mathbf{A})^{-1/2} v$. Correspondingly, the total volume of the transformed system is $V' = Nv' = (\det \mathbf{A})^{-1/2} V$. The elementary cells in $\mathbf{k}'$ space are also parallelepipeds. In three dimensions the transformed lattice is a triclinic lattice. The corresponding lattice in $\mathbf{k}'$ space is also a triclinic lattice. In two dimensions the shape of the transformed system is a parallelogram (figure 1(*c*)) which in special cases may become a rhombus (see below). Periodic boundary conditions are maintained in the transformed lattice along non-rectangular directions.

In order to obtain the transformed Hamiltonian in the standard form of a φ^4 model we need the following further transformations $\varphi'_j = (\det \mathbf{A})^{1/4} \varphi_j$, $u'_0 = (\det \mathbf{A})^{-1/2} u_0$ and

$h' = (\det \mathbf{A})^{1/4} h$. In terms of the Fourier transform $\hat{\varphi}'(\mathbf{k}') = v' \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\mathbf{x}'_j} \varphi'_j$ the transformed Hamiltonian reads

$$H^{\text{iso}}_{\text{parallel}} = V'^{-1} \sum_{\mathbf{k}'} \frac{1}{2} [r_0 + \delta\hat{K}(\mathbf{U}^{-1}\boldsymbol{\lambda}^{-1/2}\mathbf{k}')] \hat{\varphi}'(\mathbf{k}')\hat{\varphi}'(-\mathbf{k}') - h'\hat{\varphi}'(\mathbf{0})$$
$$+ u'_0 V'^{-3} \sum_{\mathbf{k}'\mathbf{p}'\mathbf{q}'} [\hat{\varphi}'(\mathbf{k}')\hat{\varphi}'(\mathbf{p}')][\hat{\varphi}'(\mathbf{q}')\hat{\varphi}'(-\mathbf{k}' - \mathbf{p}' - \mathbf{q}')] \tag{8}$$

where now $\delta\hat{K}(\mathbf{U}^{-1}\boldsymbol{\lambda}^{-1/2}\mathbf{k}') = \sum_{\alpha=1}^{d} k'^2_\alpha + O(k'^4)$ has an isotropic small-$\mathbf{k}'$ behaviour. It is understood that $H^{\text{iso}}_{\text{parallel}}$ is not exactly isotropic, but only in the asymptotic sense of long-wavelength phenomena at $O(k'^2)$. Note that our couplings $K_{i,j}$ introduced in equation (1) are *invariant* under the transformation defined above, unlike the *dimensionless* couplings $J_{i,j}$ used in previous formulations of the φ^4 lattice model [2, 8, 9]. Also the temperature variable $r_0(T)$ remains invariant. Therefore, the transformed Hamiltonian reads in real space

$$H^{\text{iso}}_{\text{parallel}} = v' \left[\sum_{i=1}^{N} \left(\frac{r_0}{2} \varphi'^2_i + u'_0 (\varphi'^2_i)^2 - h'\varphi'_i \right) + \sum_{i,j=1}^{N} \frac{K_{i,j}}{2} (\varphi'_i - \varphi'_j)^2 \right] \tag{9}$$

with the same couplings $K_{i,j}$ as in equation (1).

In summary there exists the exact relation between the lattice Hamiltonians

$$H^{\text{aniso}}_{\text{box}}(r_0, h, u_0, \mathbf{A}) = H^{\text{iso}}_{\text{parallel}}(r_0, h', u'_0) \tag{10}$$

where we have identified the lattice structure and geometry of $H^{\text{iso}}_{\text{parallel}}$ in terms of the original anisotropy matrix $\mathbf{A}$. The ensuing partition functions Z and free energies $F = -\ln Z$ (divided by $k_B T$) are exactly related by

$$Z^{\text{aniso}}_{\text{box}}(r_0, h, u_0, \mathbf{A}) = \prod_{i=1}^{N} \int \mathrm{d}^n \varphi_i \exp\left(-H^{\text{aniso}}_{\text{box}}\right)$$
$$= (\det \mathbf{A})^{-nN/4} Z^{\text{iso}}_{\text{parallel}}(r_0, h', u'_0), \tag{11}$$

$$F^{\text{aniso}}_{\text{box}}(r_0, h, u_0, \mathbf{A}) = F^{\text{iso}}_{\text{parallel}}(r_0, h', u'_0) + (nN/4) \ln(\det \mathbf{A}). \tag{12}$$

Equation (12) implies that the susceptibility

$$\chi^{\text{aniso}}_{\text{box}} = -\frac{1}{V} \frac{\partial^2 F^{\text{aniso}}_{\text{box}}}{\partial h^2} = -\frac{1}{V'} \frac{\partial^2 F^{\text{iso}}_{\text{parallel}}}{\partial h'^2} = \chi^{\text{iso}}_{\text{parallel}} \tag{13}$$

and the Binder cumulant

$$U^{\text{aniso}}_{\text{box}} = \frac{1}{3} \frac{\partial^4 F^{\text{aniso}}_{\text{box}} / \partial h^4}{\left(\partial^2 F^{\text{aniso}}_{\text{box}} / \partial h^2\right)^2} = \frac{1}{3} \frac{\partial^4 F^{\text{iso}}_{\text{parallel}} / \partial h'^4}{\left(\partial^2 F^{\text{iso}}_{\text{parall}} / \partial h'^2\right)^2} = U^{\text{iso}}_{\text{parallel}} \tag{14}$$

are invariant under the transformation defined above.

As far as the asymptotic critical behaviour is concerned we conclude that the class of finite-size scaling functions of anisotropic rectangular box systems is identical with the class of finite-size scaling functions of isotropic non-rectangular parallelepiped systems. Nevertheless, the knowledge of the latter is not sufficient to predict the finite-size effects of the anisotropic systems unless one knows the d nonuniversal eigenvalues λ_α and the nonuniversal directions of the principles axes $\mathbf{e}^{(\alpha)}$ of the anisotropic systems. In general, this requires the knowledge of $d(d+1)/2 + 1$ nonuniversal parameters.

If one accepts the Privman–Fisher hypothesis [10] of two-scale factor universality for *isotropic* confined systems, then a further consequence of equations (10)–(12) is as follows. All finite-size scaling functions of the anisotropic box system are directly determined by the

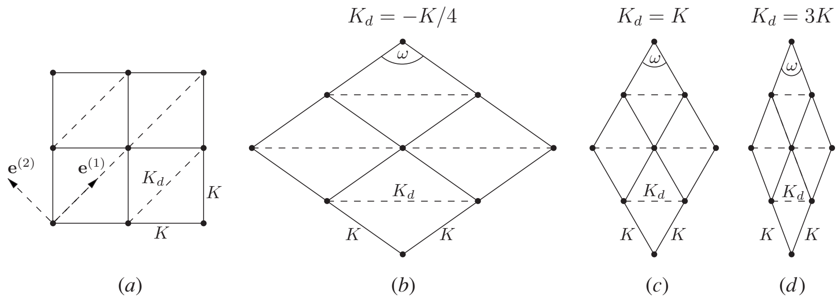

Figure 2. (*a*) Square lattice with isotropic NN couplings K and anisotropic NNN coupling K_d. The transformed lattices (*b*), (*c*), (*d*) with isotropic second moments $A'_{\alpha\beta} = \delta_{\alpha\beta}$ have the angles $\omega^{(b)} = 2\arctan(\sqrt{2})$, $\omega^{(c)} = \pi/3$, $\omega^{(d)} = 2\arctan(1/\sqrt{7})$.

universal finite-size scaling functions of the isotropic parallelepiped system where the latter depend on the geometry, but are independent of u_0 and of the microscopic lengths a_α. Thus also the scaling functions of the box system, albeit dependent on the anisotropy moments $A_{\alpha\beta}$, are independent of u_0 and a_α and of higher order couplings (such as those of φ^6 terms etc). Furthermore, they are independent of the fourth-order moments $B_{\alpha\beta\gamma\delta}$ etc. We consider this as a kind of *restricted universality*. This characterization applies also to the Binder cumulant $U(w)$ and the Casimir amplitude $\Delta(w)$ considered recently [2]. Our present analysis suggests that they are the same, e.g. for the anisotropic φ^4 field theory and for anisotropic fixed-length spin models on lattices if the geometry and boundary conditions are the same and if both models have the same second moments $A_{\alpha\beta}$. Nevertheless, we maintain that their scaling functions are not universal in the traditional sense since they do depend on the microscopic coupling constants $K_{i,j}$ entering the second moments $A_{\alpha\beta}$.

Our analysis can, of course, be generalized to other lattice structures, geometries and boundary conditions and to correlation functions. For example, for an anisotropic system in a sphere of diameter d (e.g., with free boundary conditions) the corresponding isotropic system is an ellipsoid with principle diameters $\lambda_\alpha^{-1/2} d$ in the direction of the eigenvectors $\mathbf{e}^{(\alpha)}$. As far as the calculation of the asymptotic critical behaviour is concerned it suffices, of course, to replace the lattice Hamiltonians of equation (10) by their continuum approximations with second-order gradient terms.

In the following illustrations we assume a φ^4 model on a d-dimensional simple-cubic lattice with lattice constant a in a cube of volume $V = L^d$. First we consider $d = 2$ dimensions with isotropic nearest-neighbour (NN) couplings $K_x = K_y = K$ and an anisotropic next-nearest-neighbour (NNN) coupling K_d in the diagonal $\pm(1, 1)$ directions, without a NNN coupling in the $\pm(1, -1)$ directions (figure 2(*a*)). This is the $d = 2$ version of the three-dimensional φ^4 model introduced in [2]. The anisotropy matrix is

$$(A_{\alpha\beta}) = c_0 \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix} \qquad (15)$$

where $c_0 = 2a^2(K + K_d)$ and $s = K_d/(K + K_d)$ with the eigenvalues $\lambda_{1,2} = c_0(1 \pm s)$ and the eigenvectors $\mathbf{e}^{(1)} = 2^{-1/2}(1, 1)$, $\mathbf{e}^{(2)} = 2^{-1/2}(-1, 1)$. (Because of the symmetric choice of the couplings $K_x = K_y$ the directions of the eigenvectors for $K_d \neq 0$ coincide with those of the diagonals, see figure 2(*a*).) We assume $\lambda_1 = 2a^2(K + 2K_d) > 0$, $\lambda_2 = 2a^2 K > 0$ which requires $K > 0$, $K_d > -K/2$. Application of our transformation to the $L \times L$ square yields a rhombus whose edges have the length

$$L_{\text{rhomb}} = [(2\lambda_1)^{-1} + (2\lambda_2)^{-1}]^{1/2} L = [1 + (1 + 2K_d/K)^{-1}]^{1/2} (4a^2 K)^{-1/2} L \qquad (16)$$

and the angle

$$\omega = 2\arctan[(\lambda_2/\lambda_1)^{1/2}] = 2\arctan[(1+2K_d/K)^{-1/2}]. \tag{17}$$

The lattice of the rhombus has changed second moments $A'_{\alpha\beta} = \delta_{\alpha\beta}$ such that the interaction is isotropic at $O(k'^2)$. The lattice structure and shape of the transformed φ^4 model is illustrated in figures 2(*b*)–(*d*) for the examples $\lambda_2 = 1$ and $\lambda_1 = 1/2, 3, 7$ corresponding to $K_d = -K/4, K_d = K, K_d = 3K$, respectively. Our prediction $U^{\mathrm{aniso}}_{\mathrm{square}} = U^{\mathrm{iso}}_{\mathrm{rhomb}}$ means that the Binder cumulant of the anisotropic φ^4 model on the square lattice (figure 2(*a*)) is the same as that on the corresponding transformed lattices, with the lengths and angles given by equations (16) and (17). In particular, this prediction applies to the critical Binder cumulant at $T = T_c$ and $h = 0$ for $L \to \infty$ and $L_{\mathrm{rhomb}} \to \infty$ at fixed angle ω. These predictions can be tested by MC simulations for the φ^4 lattice model. By universality, our predictions should apply also to spin models. Thus, for example for $n = 1$, we expect that the MC data shown in figure 4 of [3] should be reproduced by MC data for the family of Ising models on the transformed lattices with varying ratio K_d/K and with the shape of a rhombus where the angle ω varies between $\omega = \pi/2$ and $\omega = 2\arctan(1/3)$ corresponding to $K_d = 0$ and $K_d = 4K$, respectively.

The case (*c*) in figure 2 is an 'isotropic' triangular lattice. The critical Binder cumulant for the NN Ising model on this lattice was determined by MC simulations of Kamieniarz and Blöte [11]. They interpreted the difference with the critical Binder cumulant on the NN square lattice (with $K_x = K_y = K, K_d = 0$) as a shape effect. This interpretation for these special cases is consistent with our analysis since both models (square lattice with $K_x = K_y = K, K_d = 0$ and triangular lattice with equal couplings) have a diagonal anisotropy matrix with equal diagonal elements corresponding to isotropic systems with different shapes. In [11] no models corresponding to a nondiagonal anisotropy matrix were considered.

We briefly extend our predictions to the three-dimensional sc φ^4 lattice model of [2] with isotropic NN couplings $K_x = K_y = K_z \equiv K$ and an anisotropic NNN coupling K_d (denoted by $J/\tilde{a}^2$ and $J'/\tilde{a}^2$ in [2]). The three eigenvectors are $\mathbf{e}^{(1)} = 3^{-1/2}(1,1,1)$, $\mathbf{e}^{(2)} = 2^{-1/2}(-1,1,0)$, $\mathbf{e}^{(3)} = 6^{-1/2}(1,1,-2)$ with the eigenvalues $\lambda_1 = 2a^2(K+4K_d)$, $\lambda_2 = \lambda_3 = 2a^2(K+K_d)$. This determines uniquely the transformation of the original cube with lattice points $\mathbf{x}_j$ to a parallelepiped with lattice points $\mathbf{x}'_j = \boldsymbol{\lambda}^{-1/2}\mathbf{U}\mathbf{x}_j$. Thus we predict that the curves for the critical Binder cumulant calculated for anisotropic cubes (figure 1 of [2]) are the same as for the parallelepipeds defined above. This prediction can be tested by MC simulations for the φ^4 model and, by universality, also for spin models.

We also mention a simple application in the context of the large-n limit of the anisotropic φ^4 *continuum* model in a d-dimensional cubic geometry. In this limit the susceptibility $\chi^{\mathrm{aniso}}_{\mathrm{cube}}$ is determined by [2] $\left(\chi^{\mathrm{aniso}}_{\mathrm{cube}}\right)^{-1} = r_0 + 4u_0 n L^{-d}\sum_{\mathbf{k}}\left[\left(\chi^{\mathrm{aniso}}_{\mathrm{cube}}\right)^{-1} + \mathbf{k}\cdot\mathbf{A}\mathbf{k}\right]^{-1}$. On the other hand, we have found from *isotropic* φ^4 field theory parallel to [9] that the susceptibility $\chi^{\mathrm{iso}}_{\mathrm{parallel}}$ in a parallelepiped of volume V' with a coupling u'_0 is determined by $\left(\chi^{\mathrm{iso}}_{\mathrm{parallel}}\right)^{-1} = r_0 + 4u'_0 n V'^{-1}\sum_{\mathbf{k}'}\left[\left(\chi^{\mathrm{iso}}_{\mathrm{parallel}}\right)^{-1} + \mathbf{k}'\cdot\mathbf{k}'\right]^{-1}$, where $\mathbf{k}'$ are the wave vectors appropriate for periodic boundary conditions at the surfaces of V'. Using $u'_0 = (\det\mathbf{A})^{-1/2}u_0$, $V' = (\det\mathbf{A})^{-1/2}L^d$ and $\mathbf{k}' = \boldsymbol{\lambda}^{-1/2}\mathbf{U}\mathbf{k}$ we see that indeed $\chi^{\mathrm{aniso}}_{\mathrm{cube}} = \chi^{\mathrm{iso}}_{\mathrm{parallel}}$, in agreement with equation (13). Furthermore, the scaling function g_{cube} of the anisotropic cube of [2] can be interpreted as the scaling function $g^{\mathrm{iso}}_{\mathrm{parallel}}$ of the isotropic parallelepiped.

Finally, we briefly turn to the question whether and to what extent anisotropy may affect the asymptotic bulk critical behaviour. We focus our considerations on the standard universal bulk amplitude relations summarized in equations (2.45)–(2.54) of [1] which, according to two-scale factor universality, contain only two independent nonuniversal amplitudes. We have found that it is necessary to distinguish between two types of amplitude relations: (i) those that involve the correlation length, and (ii) those that do not involve it. We have found that the relations of the latter type (ii) are not affected by the anisotropy matrix $A_{\alpha\beta}$, but those of the former type (i) are not valid in their traditional form [1], for anisotropic systems of noncubic symmetry, because there exists no single unique correlation length for such anisotropic systems. Here we employ the principle correlation lengths [2] $\xi^{(\alpha)}$, $\alpha = 1, 2, \ldots, d$, in order to extend the validity of the relations of type (i) to anisotropic systems.

Consider, for isotropic systems, the universal relations $(\chi_0/\chi_c)(\xi_c/\xi_0)^{2-\eta} = Q_2$ and $\widehat{D}_\infty \xi_0^{2-\eta}/\chi_0 = Q_3$ proposed by Tarko and Fisher [7]. The amplitudes are defined as follows: χ_0 and ξ_0 are the asymptotic amplitudes of the bulk susceptibility $\chi \approx \chi_0 t^{-\gamma}$, $t = (T - T_c)/T_c$, and of the bulk correlation length $\xi \approx \xi_0 t^{-\nu}$ above T_c at $h = 0$, χ_c and ξ_c are their amplitudes at $T = T_c$ for small $h \neq 0$, i.e. $\chi(t = 0, h) \approx \chi_c |h|^{-(\delta-1)/\delta}$, $\xi(t = 0, h) \approx \xi_c |h|^{-\nu/\beta\delta}$, and $\widehat{D}_\infty$ is the asymptotic (small k) amplitude of the Fourier transform $\widehat{G}(\mathbf{k}) \approx \widehat{D}_\infty/k^{2-\eta}$ of the bulk order-parameter correlation function $G(\mathbf{x})$ at T_c and $h = 0$. The large-$\mathbf{x}$ behaviour of the latter is $G(\mathbf{x}) \approx D_\infty |\mathbf{x}|^{-d+2-\eta}$ where $D_\infty/\widehat{D}_\infty$ is universal. Assuming two-scale factor universality for isotropic systems we have derived the following universal relations for the anisotropic bulk systems,

$$(\chi_0/\chi_c)\left[\left(\prod_{\alpha=1}^{d} \xi_c^{(\alpha)}\right) \Big/ \left(\prod_{\alpha=1}^{d} \xi_0^{(\alpha)}\right)\right]^{(2-\eta)/d} = Q_2(d, n) = \text{universal}, \qquad (18)$$

$$\lim_{|\mathbf{x}^{(\beta)}| \to \infty} \left\{ G(\mathbf{x}^{(\beta)}) \left(\frac{|\mathbf{x}^{(\beta)}|}{\xi_0^{(\beta)}}\right)^{d-2+\eta} \frac{\prod_{\alpha=1}^{d} \xi_0^{(\alpha)}}{\chi_0} \right\} = \widetilde{Q}_3(d, n) = \text{universal}, \qquad (19)$$

where Q_2 and $\widetilde{Q}_3 = (D_\infty/\widehat{D}_\infty)Q_3$ are the same universal quantities for both isotropic and anisotropic systems within the same (d, n) universality class. Equation (19) consists of d equations for each $\beta = 1, 2, \ldots, d$, where $G(\mathbf{x}^{(\beta)})$ denotes the correlation function of the anisotropic system with the spatial argument $\mathbf{x}^{(\beta)} = x\,\mathbf{e}^{(\beta)}$ in the direction of the principle axis β. A complete description of the quantities on the left-hand sides of equations (18) and (19) for a specific system requires, in general, the knowledge of $d(d+1)/2 + 1$ nonuniversal parameters, thus up to seven parameters in three dimensions, rather than only two parameters for isotropic systems. We consider this to be a significant restriction of universality of bulk critical phenomena for a large subclass of anisotropic systems within a given (d, n) universality class.

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